Model Theory – Lecture 7 – Types and Saturated Models 2

Next lecture is on monday!

Idea: a saturated model is a "fat model" where we have witnesses for all "finite stuff"

we will not see any other ordinal here

Definition. A "(ω-) saturated model" is a model $\mathcal{M}$ that realizes all 1-types with finite ($<\omega$) parameters in $\mathcal{M}$

Examples i) the field $(\mathbb{R}, 0, 1, +, \cdot, <)$ of real numbers is <u>not</u> saturated. For instance, $(x > n)_{n \in \mathbb{N}}$ is not realized

Notice, the same type does <u>not</u> prove $(\mathbb{R}, 0, +, \cdot, <)$ is not saturated.

ii) $(\mathbb{R}, <)$ is saturated.

<u>Theorem</u> every model can be embedded in a saturated one

<u>Proof</u> Let M be a model and consider $\varphi_M \cup \{t \mid t \text{ is a 1-type}\}$

Since this is finally satisfiable, compactness grants a model M_1 and $M \hookrightarrow M_1$ in a natural way. We cannot say M_1 is saturated, as it only contains types from M in general.

We construct M_2 saturating M_1, and so on

Define $M_\omega = \bigcup_{n \in \omega} M_n$. It is easy to observe that M_ω is a model and satisfies the theorem, as it is ω-saturated (whenever a finite set of parameters is chosen it lays in one of the M_n's)

$$M \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow \cdots$$

∎

<u>Remark</u> If M is ω-saturated, then it realizes all n-types with finite parameters

We discuss the meaning of three theorems we will prove later

- A $^{(\omega-)}$saturated model is strongly $(\omega-)$ homogeneous (A)
  "elements of the same type are conjugated"
- A $(\omega-)$ saturated model is $(\omega-)$ universal (B)
  "is fat"
- Countable $\overset{\lambda}{} \; (\overset{\lambda-}{\omega-})$ saturated models are unique up to isomorphism (C)

Let's talk about homogeneity. We want to show that a partial isomorphism can be used to build a "global" isomorphism

<u>Proposition</u> Let M, N be models of our theory, N saturated. Let $\overbrace{A \cup \{a\}}^{\text{finite}} \subseteq |M|, B \subseteq |N|$ be such that $A \equiv_p B$, then there exists a b in $|N|$ such that $A \cup \{a\} \equiv B \cup \{b\}$

<u>Proof</u> Define

$\Sigma_1'(x, B)$ = all the formulas with parameters in B,

$\Sigma_1'(x, A)$ = all the formulas with parameters in A

Then, the partial isomorphism saying $A \equiv B$ proves these sets are essentially the same. We use the saturation of N to grant the existance of a witness of $\Sigma(x, B)$ and send a to it

<u>Theorem</u> Saturated models of the same cardinality are unique up to isomorphism

<u>Proof</u> Follows from Scott's theorem and the previous result, the case for ω-saturated of cardinality $\aleph_0$ (This is the only claim we care for here) ⊠

<u>Remark</u> The construction of M_ω we made before doesn't say anything about the cardinality of M_ω. It is provable that, if M is countable, then M_ω is countable as well when the language is finite/countable. (Apply RSD either to M_ω or to every M_n)

(Notice this is a particular case of the universality result (B))

Theorem (A) Let $\mathcal{M}$ be a countable ω-sat model with $(a_i)_i, (b_i)_i$ finite tuples with the same type. Then, there exists an automorphism that maps $(a_i)_i$ to $(b_i)_i$

Proof: We apply the previous proposition to $\mathcal{M} = \mathcal{N}$ and the partial isomorphism that sends $(a_i)_i$ to $(b_i)_i$

$\blacksquare$

Corollary. Let $\mathcal{N}$ be a countable model, $(a_i)_i, (b_i)_i$ finite tuples have the same type, then there exists an extension of $\mathcal{N}$ where $(a_i)_i$ and $(b_i)_i$ are conjugated